

Two-dimensional lattice for four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills

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We construct a lattice formulation of a mass-deformed two-dimensional $\mathcal{N} = (8, 8)$ super Yang-Mills theory with preserving two supercharges exactly. Gauge fields are represented by compact unitary link variables, and the exact supercharges on the lattice are nilpotent up to gauge transformations and $SU(2)_R$ rotations. Due to the mass deformation, the lattice model is free from the vacuum degeneracy problem, which was encountered in earlier approaches, and flat directions of scalar fields are stabilized giving discrete minima representing fuzzy S^2 . Around the trivial minimum, quantum continuum theory is obtained with no tuning, which serves a nonperturbative construction of the IIA matrix string theory. Moreover, around the minimum of k -coincident fuzzy spheres, four-dimensional $\mathcal{N} = 4$ $U(k)$ super Yang-Mills theory with two commutative and two noncommutative directions emerges. In this theory, sixteen supersymmetries are broken by the mass deformation to two. Assuming the breaking is soft, we give a scenario leading to undeformed $\mathcal{N} = 4$ super Yang-Mills on \mathbf{R}^4 without any fine tuning. As an evidence for the validity of the assumption, some computation of 1-loop radiative corrections is presented.

§1. Introduction

Nonperturbative aspects of supersymmetric Yang-Mills (SYM) theories play prominent roles in physics beyond the standard model¹⁾ as well as in superstring/M theory.^{2)–5)} However, to construct their nonperturbative formulations such as lattice is not a straightforward task because of the notorious difficulties of supersymmetry (SUSY) on lattice. So far, lattice formulations for SYM are constructed for one- and two-dimensional cases and $\mathcal{N} = 1$ pure SYM in three and four dimensions,⁶⁾ where no requirement of fine tunings due to the ultra-violet (UV) effects can be shown at least in perturbative arguments. For one-dimensional theory (matrix quantum mechanics) more powerful “non-lattice” technique⁷⁾ is applicable. (For corresponding lattice study, see Ref. 8).) For two-dimensional $\mathcal{N} = (2, 2)$ SYM, nonperturbative evidences for the lattice model presented in Ref. 9) to require no fine tuning have been given by numerical simulation for the gauge group $G = SU(2)$ in Ref. 10) and for

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$G = SU(N)$ with $N = 2, 3, 4, 5$ in Ref. 11) ^{*)}. Combining such techniques with the plane wave deformation¹⁴⁾ and the Myers effect,¹⁵⁾ three-dimensional theory can be obtained as a theory on fuzzy sphere.¹⁶⁾ Also, in the planar limit, four-dimensional theory can be obtained using a novel large- N reduction technique¹⁷⁾ inspired by the Eguchi-Kawai equivalence.¹⁸⁾ However, four-dimensional theories of extended SUSY at a finite rank of a gauge group are still out of reach.

We consider $\mathcal{N} = (8, 8)$ SYM with a mass deformation reminiscent of the “plane wave matrix string”^{19), 20)} in the next section, and construct a lattice formulation of the theory preserving two supercharges ^{**) in section 3. It is a modification of a lattice formulation by one of the authors (F. S.).^{9), 21), 22)} (For related constructions, see Refs. 23)–25).) Thanks to the deformation, it is free from the vacuum degeneracy problem encountered in Ref. 22) as well as from the problem of flat directions. In a perturbative argument, it is shown that the continuum theory is obtained without any fine tuning. If we turn off the mass parameter in the continuum theory, undeformed $\mathcal{N} = (8, 8)$ SYM in two dimensions, i.e. the IIA matrix string theory⁴⁾ is obtained. In section 4, we furthermore present an intriguing scenario leading to four-dimensional $\mathcal{N} = 4$ SYM with a finite-rank gauge group $G = U(k)$, starting with the lattice formulation of the two-dimensional theory. In order to realize the four-dimensional theory, we consider a k -coincident fuzzy sphere solution in the mass-deformed two-dimensional theory and take a large- N limit with scaling the deformation parameter appropriately. The crucial point is that one takes the continuum limit as two-dimensional theory first ^{***)} and then lifts the two-dimensional continuum theory to four dimensions using a matrix model formulation of noncommutative space. Section 5 is devoted to discuss possible future directions.}

§2. 2d continuum theory

We start with continuum $\mathcal{N} = (8, 8)$ SYM on \mathbf{R}^2 :

$$S_0 = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \left\{ F_{12}^2 + (D_\mu X^I)^2 - \frac{1}{2} [X^I, X^J]^2 + \Psi^T (D_1 + \gamma_2 D_2) \Psi + i \Psi^T \gamma_I [X^I, \Psi] \right\}, \quad (2.1)$$

where $\mu = 1, 2$, $I = 3, \dots, 10$, and $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. 16×16 gamma matrices γ_i ($i = 2, \dots, 10$) satisfy $\{\gamma_i, \gamma_j\} = -2\delta_{ij}$.

We rewrite this action by using Hermitian scalars X_i ($i = 3, 4$), $B_{\mathbf{A}}$ ($\mathbf{A} = 1, 2, 3$) and C , complex scalars ϕ_\pm , bosonic auxiliary fields $H_{\mathbf{A}}$, \tilde{H}_μ , \tilde{h}_i , and fermionic variables $\psi_{\pm\mu}$, $\rho_{\pm i}$, $\chi_{\pm\mathbf{A}}$ and η_\pm . As presented in appendix A, scalars ($B_{\mathbf{A}}, C, \phi_\pm$) and the

^{*)} Recently, Ref. 12) has shown that the model constructed in Ref. 13) is free from the sign problem and gives the same physics as that in Ref. 9) after an appropriate treatment of the overall $U(1)$ modes.

^{**) For simplicity, we focus on the gauge group $G = U(N)$, although the similar argument is valid also for $G = SU(N)$.}

^{***)} Similar asymmetric continuum limit is discussed on four-dimensional lattice²⁶⁾ in order to reduce the number of fine tunings.

fermionic variables are related to X_I ($I = 5, \dots, 10$) and spinor components of Ψ by a simple rearrangement, respectively. There are appropriate supercharges $Q_{\pm}^{(0)}$ by which S_0 can be written in exact form ^{*)} as

$$S_0 = Q_+^{(0)} Q_-^{(0)} \mathcal{F}^{(0)}, \quad (2.2)$$

where

$$\begin{aligned} \mathcal{F}^{(0)} = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \Big\{ & -i B_A \Phi_A - \frac{1}{3} \epsilon_{ABC} B_A [B_B, B_C] \\ & - \psi_{+\mu} \psi_{-\mu} - \rho_{+i} \rho_{-i} - \chi_{+A} \chi_{-A} - \frac{1}{4} \eta_+ \eta_- \Big\}, \end{aligned} \quad (2.3)$$

and $\Phi_1 = 2(-D_1 X_3 - D_2 X_4)$, $\Phi_2 = 2(-D_1 X_4 + D_2 X_3)$, $\Phi_3 = 2(-F_{12} + i[X_3, X_4])$. Supercharges $Q_{\pm}^{(0)}$ transform fields as

$$\begin{aligned} Q_{\pm}^{(0)} A_{\mu} &= \psi_{\pm\mu}, \quad Q_{\pm} \psi_{\pm\mu} = \pm i D_{\mu} \phi_{\pm}, \quad Q_{\mp}^{(0)} \psi_{\pm\mu} = \frac{i}{2} D_{\mu} C \mp \tilde{H}_{\mu}, \\ Q_{\pm}^{(0)} \tilde{H}_{\mu} &= [\phi_{\pm}, \psi_{\mp\mu}] \mp \frac{1}{2} [C, \psi_{\pm\mu}] \mp \frac{i}{2} D_{\mu} \eta_{\pm}, \\ Q_{\pm}^{(0)} X_i &= \rho_{\pm i}, \quad Q_{\pm}^{(0)} \rho_{\pm i} = \mp [X_i, \phi_{\pm}], \quad Q_{\mp}^{(0)} \rho_{\pm i} = -\frac{1}{2} [X_i, C] \mp \tilde{h}_i, \\ Q_{\pm}^{(0)} \tilde{h}_i &= [\phi_{\pm}, \rho_{\mp i}] \mp \frac{1}{2} [C, \rho_{\pm i}] \pm \frac{1}{2} [X_i, \eta_{\pm}], \\ Q_{\pm}^{(0)} B_A &= \chi_{\pm A}, \quad Q_{\pm}^{(0)} \chi_{\pm A} = \pm [\phi_{\pm}, B_A], \quad Q_{\mp}^{(0)} \chi_{\pm A} = -\frac{1}{2} [B_A, C] \mp H_A, \\ Q_{\pm}^{(0)} H_A &= [\phi_{\pm}, \chi_{\mp A}] \pm \frac{1}{2} [B_A, \eta_{\pm}] \mp \frac{1}{2} [C, \chi_{\pm A}], \\ Q_{\pm}^{(0)} C &= \eta_{\pm}, \quad Q_{\pm}^{(0)} \eta_{\pm} = \pm [\phi_{\pm}, C], \quad Q_{\mp}^{(0)} \eta_{\pm} = \mp [\phi_{\pm}, \phi_{\mp}], \\ Q_{\pm}^{(0)} \phi_{\pm} &= 0, \quad Q_{\mp}^{(0)} \phi_{\pm} = \mp \eta_{\pm}. \end{aligned} \quad (2.4)$$

One can see the nilpotency $(Q_+^{(0)})^2 = (Q_-^{(0)})^2 = \{Q_+^{(0)}, Q_-^{(0)}\} = 0$ up to gauge transformations.

2.1. Mass deformation

Next, we introduce a mass M to deform these charges as

$$Q_{\pm} = Q_{\pm}^{(0)} + \Delta Q_{\pm}, \quad (2.5)$$

where non-vanishing ΔQ_{\pm} transformations are

$$\begin{aligned} \Delta Q_{\pm} \tilde{H}_{\mu} &= \frac{M}{3} \psi_{\pm\mu}, \quad \Delta Q_{\pm} \tilde{h}_i = \frac{M}{3} \rho_{\pm i}, \quad \Delta Q_{\pm} H_A = \frac{M}{3} \chi_{\pm A}, \\ \Delta Q_{\pm} \eta_{\pm} &= \frac{2M}{3} \phi_{\pm}, \quad \Delta Q_{\mp} \eta_{\pm} = \pm \frac{M}{3} C. \end{aligned} \quad (2.6)$$

^{*)} This is obtained from BTFT formulation of four-dimensional $\mathcal{N} = 4$ SYM in Ref. 21) by dimensional reduction. Here, we redefine $H_A + \frac{1}{2} \epsilon_{ABC} [B_B, B_C]$, ϕ , $\tilde{\phi}$ in (4.13) in Ref. 21) as H_A , ϕ_+ , ϕ_- , respectively.

Then Q_{\pm} satisfy the anti-commutation relations,

$$Q_+^2 = \frac{M}{3}J_{++}, \quad Q_-^2 = -\frac{M}{3}J_{--}, \quad \{Q_+, Q_-\} = -\frac{M}{3}J_0, \quad (2.7)$$

up to gauge transformations, where J_0 , J_{++} and J_{--} are generators of $SU(2)_R$ symmetry acting to fields as

$$\begin{aligned} J_{++} &= \int d^2x \left[\psi_{+\mu}^\alpha(x) \frac{\delta}{\delta \psi_{-\mu}^\alpha(x)} + \chi_{+\mathbf{A}}^\alpha(x) \frac{\delta}{\delta \chi_{-\mathbf{A}}^\alpha(x)} - \eta_+^\alpha(x) \frac{\delta}{\delta \eta_-^\alpha(x)} \right. \\ &\quad \left. + 2\phi_+^\alpha(x) \frac{\delta}{\delta C^\alpha(x)} - C^\alpha(x) \frac{\delta}{\delta \phi_-^\alpha(x)} \right], \\ J_{--} &= \int d^2x \left[\psi_{-\mu}^\alpha(x) \frac{\delta}{\delta \psi_{+\mu}^\alpha(x)} + \chi_{-\mathbf{A}}^\alpha(x) \frac{\delta}{\delta \chi_{+\mathbf{A}}^\alpha(x)} - \eta_-^\alpha(x) \frac{\delta}{\delta \eta_+^\alpha(x)} \right. \\ &\quad \left. - 2\phi_-^\alpha(x) \frac{\delta}{\delta C^\alpha(x)} + C^\alpha(x) \frac{\delta}{\delta \phi_+^\alpha(x)} \right], \\ J_0 &= \int d^2x \left[\psi_{+\mu}^\alpha(x) \frac{\delta}{\delta \psi_{+\mu}^\alpha(x)} - \psi_{-\mu}^\alpha(x) \frac{\delta}{\delta \psi_{-\mu}^\alpha(x)} \right. \\ &\quad \left. + \chi_{+\mathbf{A}}^\alpha(x) \frac{\delta}{\delta \chi_{+\mathbf{A}}^\alpha(x)} - \chi_{-\mathbf{A}}^\alpha(x) \frac{\delta}{\delta \chi_{-\mathbf{A}}^\alpha(x)} + \eta_+^\alpha(x) \frac{\delta}{\delta \eta_+^\alpha(x)} - \eta_-^\alpha(x) \frac{\delta}{\delta \eta_-^\alpha(x)} \right. \\ &\quad \left. + 2\phi_+^\alpha(x) \frac{\delta}{\delta \phi_+^\alpha(x)} - 2\phi_-^\alpha(x) \frac{\delta}{\delta \phi_-^\alpha(x)} \right]. \end{aligned} \quad (2.8)$$

(α is an index for the gauge group generators.) The eigenvalues of J_0 are ± 1 for the fermions with index \pm , ± 2 for ϕ_{\pm} , and zero for the other bosonic fields. Note that ϕ_{\pm} and C form an $SU(2)_R$ triplet and each pair of $(\psi_{+\mu}, \psi_{-\mu})$, $(\chi_{+\mathbf{A}}, \chi_{-\mathbf{A}})$, $(\eta_+, -\eta_-)$ and (Q_+, Q_-) forms a doublet. In particular,

$$[J_{\pm\pm}, Q_{\pm}] = 0, \quad [J_{\pm\pm}, Q_{\mp}] = Q_{\pm}, \quad [J_0, Q_{\pm}] = \pm Q_{\pm}. \quad (2.9)$$

Using the modified supercharges, we can define Q_{\pm} -closed action as ^{*)}

$$S = \left(Q_+ Q_- - \frac{M}{3} \right) \mathcal{F}, \quad (2.10)$$

where

$$\mathcal{F} = \mathcal{F}^{(0)} + \Delta\mathcal{F}, \quad \Delta\mathcal{F} = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \left(\sum_{\mathbf{A}=1}^3 \frac{a_{\mathbf{A}}}{2} B_{\mathbf{A}}^2 + \sum_{i=3}^4 \frac{c_i}{2} X_i^2 \right). \quad (2.11)$$

That the action (2.10) is Q_{\pm} -closed can easily be seen by using (2.7), (2.9) and $SU(2)_R$ invariance of \mathcal{F} . After integrating out the auxiliary fields, $B_{\mathbf{A}}$ and X_i have positive mass terms as long as the parameters $a_{\mathbf{A}}$ and c_i all lie in the open interval

^{*)} This kind of deformation is extended to various SYM theories in Ref. 27).

$(-2M/3, 0)$. Here we take $a_1 = a_2 = a_3 = -\frac{2M}{9}$ and $c_3 = c_4 = -\frac{4M}{9}$ for convenience. Then the action reads

$$S = S_0 + \Delta S, \quad (2.12)$$

where

$$\begin{aligned} \Delta S = \frac{1}{g_{2d}^2} \int d^2x \operatorname{Tr} \Big\{ & \frac{2M^2}{81} (B_{\mathbf{A}}^2 + X_i^2) + \frac{M^2}{9} \left(\frac{C^2}{4} + \phi_+ \phi_- \right) - \frac{M}{2} C[\phi_+, \phi_-] \\ & + \frac{2M}{3} \psi_{+\mu} \psi_{-\mu} + \frac{2M}{9} \rho_{+i} \rho_{-i} + \frac{4M}{9} \chi_{+A} \chi_{-A} - \frac{M}{6} \eta_+ \eta_- \\ & - \frac{4iM}{9} B_3 (F_{12} + i[X_3, X_4]) \Big\}. \end{aligned} \quad (2.13)$$

From this expression one can see some similarity to the plane wave matrix model¹⁴ and to PP wave matrix strings.¹⁹ The first two terms in the first line and the terms in the second line in (2.13) give mass terms to scalars and to fermions, respectively. The third term represents the so called Myers term.¹⁵ Thanks to these terms, fuzzy S^2 configurations satisfying

$$[\phi_+, \phi_-] = \frac{M}{3} C, \quad [C, \phi_{\pm}] = \pm \frac{2M}{3} \phi_{\pm}, \quad B_{\mathbf{A}} = X_i = 0 \quad (2.14)$$

give the minima of the action ($S = 0$) preserving Q_{\pm} SUSYs. Note that the last term in (2.13) is purely imaginary. Also, we should recognize that the mass-deformed action preserves only two supercharges (Q_{\pm}) but the other 14 charges are softly broken by the deformation.

§3. Lattice formulation

In this section we put the deformed theory on a two-dimensional square lattice. We use link variables $U_{\mu} = e^{iaA_{\mu}}$ belonging to the gauge group $U(N)$, where a is the lattice spacing. Other lattice fields, defined on sites, are made dimensionless by multiplying suitable powers of a to the continuum counterparts:

$$\begin{aligned} (\text{scalars})^{\text{lat}} &= a (\text{scalars})^{\text{cont}}, & (\text{fermions})^{\text{lat}} &= a^{3/2} (\text{fermions})^{\text{cont}}, \\ Q_{\pm}^{\text{lat}} &= a^{1/2} Q_{\pm}^{\text{cont}}. \end{aligned} \quad (3.1)$$

Also, dimensionless coupling constants on the lattice are

$$g_0 = ag_{2d}, \quad M_0 = aM. \quad (3.2)$$

The supersymmetry transformations are realized as

$$\begin{aligned} Q_{\pm} U_{\mu}(x) &= i\psi_{\pm\mu}(x) U_{\mu}(x), \\ Q_{\pm} \psi_{\pm\mu}(x) &= i\psi_{\pm\mu}(x) \psi_{\pm\mu}(x) \pm iD_{\mu} \phi_{\pm}(x), \\ Q_{\mp} \psi_{\pm\mu}(x) &= \frac{i}{2} \{ \psi_{+\mu}(x), \psi_{-\mu}(x) \} + \frac{i}{2} D_{\mu} C(x) \mp \tilde{H}_{\mu}(x), \end{aligned}$$

$$\begin{aligned}
Q_{\pm} \tilde{H}_{\mu}(x) = & -\frac{1}{2} \left[\psi_{\mp\mu}(x), \phi_{\pm}(x) + U_{\mu}(x) \phi_{\pm}(x + \hat{\mu}) U_{\mu}(x)^{\dagger} \right] \\
& \pm \frac{1}{4} \left[\psi_{\pm\mu}(x), C(x) + U_{\mu}(x) C(x + \hat{\mu}) U_{\mu}(x)^{\dagger} \right] \\
& \mp \frac{i}{2} D_{\mu} \eta_{\pm}(x) \pm \frac{1}{4} [\psi_{\pm\mu}(x) \psi_{\pm\mu}(x), \psi_{\mp\mu}(x)] \\
& + \frac{i}{2} [\psi_{\pm\mu}(x), \tilde{H}_{\mu}(x)] + \frac{M_0}{3} \psi_{\pm\mu},
\end{aligned} \tag{3.3}$$

for the lattice fields $U_{\mu}, \psi_{\pm\mu}$ and \tilde{H}_{μ} . D_{μ} is a covariant forward difference operator defined by

$$D_{\mu} A(x) \equiv U_{\mu}(x) A(x + \hat{\mu}) U_{\mu}(x)^{\dagger} - A(x), \tag{3.4}$$

for any adjoint field $A(x)$. Transformation of the other fields is the same as the one in continuum with the obvious replacement $M \rightarrow M_0$. Then the anti-commutation relation (2.7) holds on the lattice with $M \rightarrow M_0$. In order to construct a corresponding lattice action, we take lattice counterparts of $\Phi_{\mathbf{A}}$ as

$$\begin{aligned}
\Phi_1(x) &= 2(-D_1 X_3(x) - D_2 X_4(x)), \\
\Phi_2(x) &= 2(-D_1^* X_4(x) + D_2^* X_3(x)), \\
\Phi_3(x) &= \frac{i(U_{12}(x) - U_{21}(x))}{1 - \epsilon^{-2} \|1 - U_{12}(x)\|^2} + 2i[X_3(x), X_4(x)],
\end{aligned} \tag{3.5}$$

where D_{μ}^* is a covariant backward difference operator,

$$D_{\mu}^* A(x) \equiv A(x) - U_{\mu}(x - \hat{\mu})^{\dagger} A(x - \hat{\mu}) U_{\mu}(x - \hat{\mu}), \tag{3.6}$$

$U_{\mu\nu}(x) = U_{\mu}(x) U_{\nu}(x + \hat{\mu}) U_{\mu}(x + \hat{\nu})^{\dagger} U_{\nu}(x)^{\dagger}$ is a plaquette variable, ϵ is a positive constant satisfying $0 < \epsilon < 2$, and the norm of a matrix is defined by $\|A\| = \sqrt{\text{Tr}(AA^{\dagger})}$. The first term of the r.h.s. of $\Phi_3(x)$ is a lattice counterpart of the field strength F_{12} . It is the same as the situation in the lattice formulation for two-dimensional $\mathcal{N} = (2, 2)$ $U(N)$ SYM in Ref. 9). Q_{\pm} -invariant lattice action is given as

$$S_{\text{lat}} = \left(Q_+ Q_- - \frac{M_0}{3} \right) \mathcal{F}_{\text{lat}} \tag{3.7}$$

with \mathcal{F}_{lat} being the same form as \mathcal{F} in (2.11) under the trivial replacement $\frac{1}{g_{2d}^2} \int d^2x \rightarrow \frac{1}{g_0^2} \sum_x$, $M \rightarrow M_0$, when the admissibility condition $\|1 - U_{12}(x)\| < \epsilon$ is satisfied for $\forall x$. Otherwise, $S_{\text{lat}} = +\infty$.

Note that in Eqs. (3.5) the covariant forward difference is used for Φ_1 , while the covariant backward difference is used for Φ_2 . With this choice, no species doubler appears in both of bosonic and fermionic kinetic terms. Note also that the fuzzy sphere solution of the lattice version of the equations (2.14)

$$C = \frac{2M_0}{3} L_3, \quad \phi_{\pm} = \frac{M_0}{3} (L_1 \pm iL_2), \quad B_{\mathbf{A}} = X_i = 0 \tag{3.8}$$

preserves Q_{\pm} SUSYs at regularized level, where L_a ($a = 1, 2, 3$) belong to an N -dimensional representation of $SU(2)$ generators satisfying $[L_a, L_b] = i\epsilon_{abc} L_c$.

3.1. No unphysical degenerate minima

Here, we will check that the lattice action has the minimum only at the pure gauge configuration $U_{12}(x) = \mathbb{1}_N$, which guarantees that the weak field expansion $U_\mu(x) = 1 + iaA_\mu(x) + \frac{(ia)^2}{2!}A_\mu(x)^2 + \dots$ is allowed in the continuum limit so that the lattice theory converges to the desired continuum theory at the classical level.

After integrating out the auxiliary fields, bosonic part of the action S_{lat} takes the form

$$S_{\text{lat}}^{(B)} = \frac{1}{g_0^2} \sum_x \text{tr} \left[\frac{2M_0^2}{81} (X_i(x)^2 + B_{\mathbf{A}}(x)^2) - i \frac{2M_0}{9} B_3(x) \Phi_3^{(-)}(x) \right] + S_{\text{PDT}} \quad (3.9)$$

where $\Phi_3^{(-)}(x)$ is $\Phi_3(x)$ in (3.5) with the sign of the first term flipped, and S_{PDT} denotes positive (semi-)definite terms. We will treat the second term, which is purely imaginary, as an operator in the reweighting method, and consider the minimum of the remaining part of $S_{\text{lat}}^{(B)}$. The mass terms in (3.9) fix the minimum at

$$X_i(x) = B_{\mathbf{A}}(x) = 0, \quad (3.10)$$

which is independent of S_{PDT} . At the minimum (3.10), S_{PDT} becomes

$$\begin{aligned} S_{\text{PDT}} = & \frac{1}{g_0^2} \sum_x \text{tr} \left[\sum_\mu (D_\mu X_p(x))^2 + \left(i[X_p(x), X_q(x)] + \frac{M_0}{3} \epsilon_{pqr} X_r(x) \right)^2 \right] \\ & + \frac{1}{4g_0^2} \sum_x \frac{\text{tr} [-(U_{12}(x) - U_{21}(x))^2]}{\left(1 - \frac{1}{\epsilon^2} \|1 - U_{12}(x)\|^2\right)^2} \end{aligned} \quad (3.11)$$

with $C = 2X_8$, $\phi_\pm = X_9 \pm iX_{10}$ and $p, q, r = 8, 9, 10$. Since the last term representing the gauge kinetic term is the same as in the case of two-dimensional $\mathcal{N} = (2, 2)$ SYM discussed in Ref. 9), the admissibility condition with $0 < \epsilon < 2$ for the gauge group $U(N)$ singles out the trivial minimum $U_{12}(x) = \mathbb{1}_N$. It shows that the lattice action has a stable physical vacuum and unphysical degeneracies of vacua seen in the former formulation²²⁾ do not appear. The mass deformation preserving Q_\pm SUSYs is crucial to stabilize flat directions of scalars as well as to remove the unphysical minima for gauge fields.

3.2. No need of fine tuning

Next, we discuss in the perturbation theory that the desired quantum continuum theory is obtained without any fine tuning.

In the theory near the continuum limit with the auxiliary fields integrated out, let us consider local operators of the type:

$$\mathcal{O}_p(x) = M^m \varphi(x)^\alpha \partial^\beta \psi(x)^{2\gamma}, \quad p \equiv m + \alpha + \beta + 3\gamma \quad (3.12)$$

where φ denotes scalar fields as well as gauge fields, ψ fermionic fields, and ∂ derivatives. The mass dimension of \mathcal{O}_p is p , and $m, \alpha, \beta, \gamma = 0, 1, 2, \dots$. From dimensional analysis, it can be seen that radiative corrections from UV region of loop momenta

to \mathcal{O}_p have the form

$$\left(\frac{1}{g_{2d}^2} c_0 a^{p-4} + c_1 a^{p-2} + g_{2d}^2 c_2 a^p + \dots \right) \int d^2x \mathcal{O}_p(x), \quad (3.13)$$

up to possible powers of $\ln(aM)$. c_0, c_1, c_2 are dimensionless numerical constants. The first, second and third terms in the parenthesis are contributions from tree, 1-loop and 2-loop effects, respectively. The “...” is effects from higher loops, which are irrelevant for the analysis.

Since relevant or marginal operators generated by loop effects possibly appear from nonpositive powers of a in the second and third terms in (3.13), we should see operators with $p = 0, 1, 2$. They are φ , $M\varphi$ and φ^2 . (Note that $\mathbb{1}$, M , M^2 and $\partial\varphi$ are not dynamical.) Candidates for φ are linear combinations of $\text{tr } X_i$ and $\text{tr } B_A$ from gauge and $SU(2)_R$ symmetries. But, all of them are not invariant under Q_\pm SUSYs, and thus are forbidden to appear. Similarly, $M\varphi$ and φ^2 are not allowed to be generated.

Therefore, in the perturbative argument, we can conclude that any relevant or marginal operators except nondynamical operators do not appear radiatively, meaning that no fine tuning is required to take the continuum limit.

3.3. Matrix String theory

The mass-deformed $\mathcal{N} = (8, 8)$ SYM in two dimensions can be obtained from the constructed lattice theory around the trivial minimum $C = \phi_\pm = 0$ as seen in the previous section. Since M is a soft mass breaking 16 SUSYs to Q_\pm , undeformed theory, which is nothing but the IIA matrix string theory,⁴⁾ can be defined by turning off M after the continuum limit.

§4. 4d $\mathcal{N} = 4$ SYM

In this section, we discuss a scenario to obtain four-dimensional $\mathcal{N} = 4$ SYM from the lattice formulation given in the previous section.

Let us consider the lattice theory expanded around the minimum of k -coincident fuzzy S^2 given by (3.8) with

$$L_a = L_a^{(n)} \otimes \mathbb{1}_k \quad (a = 1, 2, 3) \quad \text{and} \quad N = nk. \quad (4.1)$$

$L_a^{(n)}$ are $SU(2)$ -generators of an $n(= 2j + 1)$ -dimensional irreducible representation of spin j .

First, we take the continuum limit of the two-dimensional lattice theory. Then, we obtain four-dimensional $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbf{R}^2 \times (\text{Fuzzy } S^2)$ with 16 SUSYs broken to Q_\pm by M ^{*)}. The fuzzy S^2 has the radius $R = 3/M$, and its noncommutativity (fuzziness) is characterized by the parameter $\Theta = \frac{18}{M^2 n}$. UV cutoff in the S^2 directions is set at $\Lambda = \frac{M}{3} \cdot 2j$. These properties of the fuzzy S^2 are seen by doing a similar calculation as presented in Refs. 16), 17), 28). In particular, momentum

^{*)} Due to the infinite volume of \mathbf{R}^2 , tunnelling among discrete minima of various fuzzy sphere solutions is suppressed to stabilize each fuzzy sphere background.

modes of a field, say $B_{\mathbf{A}}$, on two dimensions are expanded further by fuzzy spherical harmonics:

$$\tilde{B}_{\mathbf{A}}(q) = \sum_{J=0}^{2j} \sum_{m=-J}^J \hat{Y}_{Jm}^{(jj)} \otimes b_{\mathbf{A}, Jm}, \quad (4.2)$$

corresponding to the expression (4.1). The fuzzy spherical harmonic $\hat{Y}_{Jm}^{(jj)}$ is an $n \times n$ matrix whose elements are given by Clebsch-Gordon (C-G) coefficients as¹⁷⁾

$$\hat{Y}_{Jm}^{(jj)} = \sqrt{n} \sum_{r,r'=-j}^j (-1)^{-j+r'} C_{jrj-r'}^{Jm} |j\ r\rangle \langle j\ r'| \quad (4.3)$$

with an orthonormal basis $|j\ r\rangle$ representing $L_a^{(n)}$ in the standard way:

$$\begin{aligned} \left(L_1^{(n)} \pm i L_2^{(n)} \right) |j\ r\rangle &= \sqrt{(j \mp r)(j \pm r + 1)} |j\ r \pm 1\rangle, \\ L_3^{(n)} |j\ r\rangle &= r |j\ r\rangle, \end{aligned} \quad (4.4)$$

and the modes $b_{\mathbf{A}, Jm}$ are $k \times k$ matrices. It is seen that the fuzzy spherical harmonics are eigen-modes of the Laplacian on the fuzzy S^2 :

$$\sum_{a=1}^3 \left(\frac{M}{3} \right)^2 [L_a^{(n)}, [L_a^{(n)}, \hat{Y}_{Jm}^{(jj)}]] = \left(\frac{M}{3} \right)^2 J(J+1) \hat{Y}_{Jm}^{(jj)}, \quad (4.5)$$

giving the rotational energy with the angular momentum J on the sphere of the radius $R = 3/M$. The UV cutoff $\Lambda = \frac{M}{3} \cdot 2j$ can be read off from the upper limit of the sum of J in the expansion (4.2). The fuzzy S^2 is a two-dimensional noncommutative (NC) space, which is analogous to the phase space of some one-dimensional quantum system, and the noncommutativity Θ to the Planck constant \hbar . The quantum phase space is divided into small cells of the size $2\pi\hbar$, whose number is equal to the dimension of the Hilbert space. Correspondingly, the area of the S^2 is divided into n cells of the size $2\pi\Theta$:

$$4\pi R^2 = n \cdot 2\pi\Theta, \quad (4.6)$$

leading to the value $\Theta = \frac{18}{M^2 n}$.

Notice, differently from the two-dimensional case, it is not clear whether the SUSY breaking by M is soft, because M appears not only in mass terms in the action but also in the geometry of the fuzzy S^2 . Let us proceed assuming that the breaking is soft^{*)}. We will give some argument later for the validity of the assumption.

Next, we take successive limits by following the two steps:

- **Step 1:** Take large n limit with Θ and k fixed. Namely, $M \propto n^{-1/2} \rightarrow 0$ and $\Lambda \propto n^{1/2} \rightarrow \infty$.
- **Step 2:** Send Θ to zero.

^{*)} The assumption is plausible from the viewpoint of the mapping rule between matrix model and Yang-Mills theory on noncommutative space.²⁹⁾

4.1. Step 1

At the step 1, the fuzzy S^2 is decompactified to the NC Moyal plane \mathbf{R}_Θ^2 . From the assumption, the theory becomes $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbf{R}^2 \times \mathbf{R}_\Theta^2$ with 16 SUSYs restored. The gauge coupling constant of the four-dimensional theory is given in the form

$$g_{4d}^2 = 2\pi\Theta g_{2d}^2. \quad (4.7)$$

In the limit, the expansion (4.2) by the fuzzy spherical harmonics can be essentially transcribed to the one by plane waves on \mathbf{R}_Θ^2 :

$$\tilde{B}_\mathbf{A}(q) = \int \frac{d^2\tilde{q}}{(2\pi)^2} e^{i\tilde{q}\cdot\hat{x}} \otimes \tilde{b}_\mathbf{A}(\mathbf{q}), \quad (4.8)$$

where q and \tilde{q} are two-momenta on \mathbf{R}^2 and \mathbf{R}_Θ^2 respectively, the position operator $\hat{x} = (\hat{x}_1, \hat{x}_2)$ on \mathbf{R}_Θ^2 satisfies $[\hat{x}_1, \hat{x}_2] = i\Theta$, and $\mathbf{q} \equiv (q, \tilde{q})$ represents a four-momentum. The modes $\tilde{b}_\mathbf{A}(\mathbf{q})$ in the four-dimensional space are $k \times k$ matrices. It is easy to calculate the inner product between plane waves on \mathbf{R}_Θ^2 :

$$\text{Tr} \left(e^{i\tilde{p}\cdot\hat{x}} e^{i\tilde{q}\cdot\hat{x}} \right) = \frac{2\pi}{\Theta} \delta^2(\tilde{p} + \tilde{q}), \quad (4.9)$$

which leads to the Θ -dependence of the relation (4.7).

Let us discuss radiative corrections in four-dimensional SYM on $\mathbf{R}^2 \times (\text{Fuzzy } S^2)$. We give an argument below that there is no radiative correction which prevents from the full 16 SUSYs being restored after the step 1.

In quantum field theory defined on NC space with a constant noncommutativity, there are two kinds of Feynman diagrams. One is planar diagrams. They have no NC phase factors depending on loop momenta, and their behavior is the same as that in the corresponding theory on the ordinary space.³⁰⁾ The other is nonplanar diagrams. They have NC phase factors, which improve the UV behavior of the diagrams. But, when some of the NC phases vanish in the infra-red (IR) region of external or loop momenta, singularities may arise, whose origin is the UV singularities in the corresponding theory on the ordinary space (UV/IR mixing).³¹⁾ Therefore, we can say that UV behavior of planar and nonplanar diagrams in the theory on NC space is not worse than that in the corresponding theory on the ordinary space. Let us consider the superficial degree of UV divergences of Feynman diagrams in ordinary four-dimensional gauge theory:

$$D = 4 - E_B - \frac{3}{2}E_F, \quad (4.10)$$

where E_B (E_F) is the number of the external lines of bosons (fermions). In our case, the divergence of $D = 3$, that is from $E_B = 1$, is absent since the operator φ is forbidden by Q_\pm SUSYs as in the two-dimensional case. Thus, the possible most severe divergences are of the degree $D = 2$. The leading Λ^2 terms are expected to cancel each other by 16 SUSYs under the assumption that M is soft. For radiative corrections to gauge invariant observables, divergences possibly originate from the

mass deformation, whose behavior is expected as ^{*)}

$$M^p \left(\ln \frac{\Lambda}{M} \right)^q = \mathcal{O}(M^p (\ln n)^q) \quad (p = 1, 2, q = 1, 2, \dots). \quad (4.11)$$

However, such terms disappear in the limit of the step 1.

Hence, there appears no radiative correction preventing restoration of the full SUSYs after the step 1, leading to $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbf{R}^2 \times \mathbf{R}_\Theta^2$ with 16 supercharges.

4.2. Step 2

In four-dimensional $\mathcal{N} = 4$ SYM on NC space, the commutative limit ($\Theta \rightarrow 0$) is believed to be smooth,^{34), 35)} that is, desired $\mathcal{N} = 4$ $U(k)$ SYM on usual flat \mathbf{R}^4 should be obtained with no fine tuning after the step 2.

4.3. Check of the scenario

As a check of the scenario presented in the above, we computed 1-loop radiative corrections to scalar kinetic terms of $B_{\mathbf{A}=1,2}$. Although details are presented in a separate publication,³⁶⁾ contribution from planar diagrams to the kinetic terms in the 1-loop effective action finally becomes

$$\frac{1}{g_{4d}^2} \int \frac{d^4 \mathbf{q}}{(2\pi)^4} \sum_{\mathbf{A}=1,2} \text{tr}_k \left[\mathbf{q}^2 \tilde{b}_{\mathbf{A}}^{(R)}(-\mathbf{q}) \tilde{b}_{\mathbf{A}}^{(R)}(\mathbf{q}) \right] \left\{ 1 + \frac{g_{4d}^2 k}{4\pi^2} \left(-\frac{1}{2} \ln \frac{\mathbf{q}^2}{\mu_R^2} + 1 \right) + \mathcal{O}(g_{4d}^4) \right\} \quad (4.12)$$

after the limit of the step 1. $\tilde{b}_{\mathbf{A}}^{(R)}(\mathbf{q})$ are renormalized momentum modes which are related to the modes $\tilde{b}_{\mathbf{A}}(\mathbf{q})$ of the bare fields $B_{\mathbf{A}}$ as

$$\tilde{b}_{\mathbf{A}}^{(R)}(\mathbf{q}) = \left(1 + \frac{g_{4d}^2 k}{4\pi^2} \ln \frac{\Lambda}{\mu_R} \right)^{1/2} \tilde{b}_{\mathbf{A}}(\mathbf{q}), \quad (4.13)$$

where μ_R is the renormalization point ^{**)} . The logarithmic nonlocal term in (4.12) has a definite physical meaning contributing to anomalous scaling dimension of the operator. Since the result does not depend on Θ , the limit of the step 2 is trivial.

Note that four-dimensional rotational symmetry is restored in (4.12), which can be regarded as an evidence of the restoration of the full 16 SUSYs and of the softness of M . Furthermore, we found that nonplanar contribution is essentially the same as the planar contribution, supporting the smoothness of the commutative limit $\Theta \rightarrow 0$.

^{*)} We should note that the behavior (4.11) is not valid for gauge-dependent divergences which can be absorbed into wave function renormalization. (4.11) is derived based on UV finiteness of the undeformed four-dimensional $\mathcal{N} = 4$ SYM,^{32), 33)} where the finiteness holds except such divergences.

^{**)} Although four-dimensional $\mathcal{N} = 4$ SYM is said to be UV finite, divergence of the wave function renormalization can appear as a gauge artifact.³²⁾ In fact, a modified light-cone gauge fixing in Refs. 32) leads to no UV divergence even in the part concerning the wave function renormalization, differently from the 1-loop computation in Feynman gauge fixing.³³⁾ The point is that even if UV divergences appear in radiative corrections, all of them can be absorbed by rescaling the fields.³³⁾ We adopted a Feynman-like gauge fixing in the calculation.

§5. Discussions

We constructed a lattice formulation of two-dimensional $\mathcal{N} = (8, 8)$ SYM with a mass deformation, which preserves two supercharges. It serves a basis of nonperturbative investigation of the IIA matrix string theory. Also, it gives an intriguing scenario to obtain four-dimensional $\mathcal{N} = 4$ $U(k)$ SYM with arbitrary k , requiring no fine tuning.

It is interesting to extend such construction to theories coupled to fundamental matters. Although it is difficult to introduce fundamental fields directly, bi-fundamental fields can easily be incorporated. For instance, let us start with a two-dimensional system with SUSYs, which is obtained by dimensional reduction from the corresponding theory in four dimensions. The action $S_0 = S_{0,g} + S_{0,g'} + S_m$ is

- $S_{0,g}$ is the action of $U(N)$ SYM with gauge field A_μ and adjoint matters X_I ($I = 1, \dots, \ell_a$):

$$S_{0,g} = \frac{1}{g_{2d}^2} \int d^2x \operatorname{tr}_N \left[F_{12}^2 + (D_\mu X_I)^2 - \frac{1}{2} [X_I, X_J]^2 + (\text{fermions}) \right] \quad (5.1)$$

with $D_\mu X_I = \partial_\mu X_I + i[A_\mu, X_I]$.

- $S_{0,g'}$ is the action of $U(N')$ SYM with gauge field A'_μ and adjoint matters X'_I ($I = 1, \dots, \ell_a$):

$$S_{0,g'} = \frac{1}{(g'_{2d})^2} \int d^2x \operatorname{tr}_{N'} \left[(F'_{12})^2 + (D_\mu X'_I)^2 - \frac{1}{2} [X'_I, X'_J]^2 + (\text{fermions}) \right] \quad (5.2)$$

with $D_\mu X'_I = \partial_\mu X'_I + i[A'_\mu, X'_I]$. It is essentially the same as (5.1) except the change $g_{2d} \rightarrow g'_{2d}$, $N \rightarrow N'$.

- S_m is the action of $U(N) \times U(N')$ bi-fundamental matters Φ_i ($i = 1, \dots, \ell_f$) coupled to the above two systems:

$$S_m = \int d^2x \operatorname{tr}_{N'} \left[(D_\mu \Phi_i)^\dagger D_\mu \Phi_i + (X_I \Phi_i - \Phi_i X'_I)^\dagger (X_I \Phi_i - \Phi_i X'_I) + (\text{fermions}) \right] \quad (5.3)$$

with $D_\mu \Phi_i = \partial_\mu \Phi_i + iA_\mu \Phi_i - i\Phi_i A'_\mu$.

We consider the situation that both of $S_{0,g}$ and $S_{0,g'}$ allow a deformation by mass M preserving some SUSYs as discussed in section 2, and that deformed actions S_g and $S_{g'}$ have classical solutions of k - and k' -coincident fuzzy S^2 :

$$\begin{aligned} X_a &= \frac{M}{3} L_a^{(n)} \otimes \mathbb{1}_k & (N = nk), \\ X'_a &= \frac{M}{3} L_a^{(n)} \otimes \mathbb{1}_{k'} & (N' = nk'), \end{aligned} \quad (5.4)$$

with all the other fields nil, respectively. (We labelled the index I so that scalars with $I = 1, 2, 3$ satisfy the fuzzy S^2 configurations.) Then, for k and k' general, vanishing the bi-fundamental fields gives the minima of the zero total action $S \equiv$

$S_g + S_{g'} + S_m = 0$. Expanding S around the background (5.4) leads to two systems of gauge and adjoint fields with gauge groups $U(k)$ and $U(k')$, which are coupled by $U(k) \times U(k')$ bi-fundamental matters. They are defined on $\mathbf{R}^2 \times (\text{Fuzzy } S^2)$, analogous to the situation seen in section 4. Finally, after turning off the coupling g'_{2d} , we obtain the system of $U(k)$ gauge and adjoint fields coupled to $k'\ell_f$ fundamental matters (with $U(k')$ gauge and adjoint fields becoming free and decoupled) on $\mathbf{R}^2 \times (\text{Fuzzy } S^2)$. Therefore, if the system of the action S can be realized on lattice, and if successive limits analogous to those discussed in section 4 can be taken safely, it is expected to obtain the desirable quantum system on \mathbf{R}^4 . (Similar construction using the Eguchi-Kawai equivalence can be found in Ref. 37).)

Using our formalism, many interesting theories will be realized on computer. We expect new insights into nonperturbative dynamics of supersymmetric theories will be obtained in near future.

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Appendix A

Notations

We give the relation between field variables in (2.1) and those in (2.2), (2.3). For scalars,

$$B_1 = -X_5, \quad B_2 = X_6, \quad B_3 = X_7, \quad C = 2X_8, \quad \phi_{\pm} = X_9 \pm iX_{10}. \quad (\text{A.1})$$

For fermionic variables,

$$\begin{aligned} \Psi &= U_{16} \Psi^{(0)}, \\ (\Psi^{(0)})^T &\equiv \left(\rho_{+3}, \rho_{+4}, \psi_{+2}, \psi_{+1}, -\chi_{+1}, \chi_{+2}, \chi_{+3}, \frac{1}{2}\eta_{+}, \right. \\ &\quad \left. \rho_{-3}, \rho_{-4}, \psi_{-2}, \psi_{-1}, -\chi_{-1}, \chi_{-2}, \chi_{-3}, \frac{1}{2}\eta_{-} \right), \end{aligned} \quad (\text{A.2})$$

where U_{16} is a 16×16 unitary matrix of the form

$$U_{16} = \frac{1}{2} \begin{pmatrix} & \mathcal{A}_{12} & \mathcal{A}_{13} & \\ \mathcal{A}_{21} & & & \\ & & & \mathcal{A}_{24} \\ & & & \end{pmatrix} \quad (\text{A.3})$$

with

$$\begin{aligned}
\mathcal{A}_{12} &\equiv \begin{pmatrix} 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 \\ 0 & 0 & -1 & -i \\ 1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}, & \mathcal{A}_{13} &\equiv \begin{pmatrix} 0 & 0 & -1 & i \\ -1 & i & 0 & 0 \\ 0 & 0 & -i & -1 \\ -i & -1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & i \\ -i & 1 & 0 & 0 \\ 0 & 0 & i & -1 \end{pmatrix}, \\
\mathcal{A}_{21} &\equiv \begin{pmatrix} 0 & 0 & i & 1 \\ i & 1 & 0 & 0 \\ 0 & 0 & -1 & i \\ -1 & i & 0 & 0 \\ i & -1 & 0 & 0 \\ 0 & 0 & -i & 1 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix}, & \mathcal{A}_{24} &\equiv \begin{pmatrix} 0 & 0 & -1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ -i & -1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ -i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{pmatrix}. \tag{A.4}
\end{aligned}$$

Also, the explicit form of the gamma matrices we used is

$$\begin{aligned}
\gamma_2 &= -i\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_3, \\
\gamma_3 &= -i\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_1, \\
\gamma_4 &= +i\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2, \\
\gamma_5 &= -i\sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_2, \\
\gamma_6 &= +i\sigma_2 \otimes \mathbb{1}_2 \otimes \sigma_1 \otimes \sigma_2, \\
\gamma_7 &= -i\sigma_2 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_2, \\
\gamma_8 &= -i\sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2, \\
\gamma_9 &= +i\mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2, \\
\gamma_{10} &= +i\sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \tag{A.5}
\end{aligned}$$

with σ_a ($a = 1, 2, 3$) being the Pauli matrices.

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